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HOMEOMORPHISMS WITH MARKOV PARTITIONS

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ABSTRACT

Necessary and sufficient conditions for expansive homeomorphisms to have Markov partitions are given.

It is known that the existence of Markov partitions suplies us important informations in dynamical systems (for examples, studies of equilibrium states 5] and zeta functions 24]).

The construction of Markov partitions first was done for Anosov diffeomorphisms by Ja. G. Sinai 35]. After that R. Bowen 5] constructed Markov partitions for nonwandering sets of Axiom A diffeomorphisms. Following 5],35], we see that the notion so called canonical coordinates plays an important role to construct Markov partitions. In topological setting, K. Hiraide 20] proved that every expansive homeomorphism with POTP has canonical coordinates and further such a homeomorphism has Markov partitions. It is known (N. Aoki 2] and 3]) that every expansive automorphism of a solenoidal group has POTP and such an automorphism has canonical coordinates. Thus the automorphism has Markov partitions (20]).

However homeomorphisms with Markov partitions do not necessarily have canonical coordinates. In fact, every pseudo-Anosov map has Markov partitions and does not have canonical coordinates (see paragraph 9 and 10 of 17]).

It is natural to ask what kind of expansive homeomorphisms have Markov partitions. The purpose of this paper is to give necessary and sufficient conditions for expansive homeomorphisms to have Markov partitions. More precisely we can describe our result as follows;

THEOREM. Let X be a compact metric space and f be an expansive self-homeomorphism of X. Then the following conditions are equivalent;

(I) there exists $c > 0$ with $2c$ is an expansive constant such that for every $x \in X$ there exists an $\eta = \eta(x) > 0$ such that $\{Y_c(y) \cap B_\eta(x) \mid y \in B_\eta(x)\}$ is finite,

(II) there exists $c > 0$ with $2c$ is an expansive constant such that for every $x \in X$ there exists a $\delta = \delta(x) > 0$ such that $\{Z_c(y) \cap B_\delta(x) \mid y \in B_\delta(x)\}$ is finite,

(III) (X,f) has TPOTP,

(IV) (X,f) has a Markov partition.

In the remainder, we shall give some definitions which are used in our theorem. The proof will appear elsewhere.

Let X be a compact metric space with metric d, and f be a self-homeomorphism of X. For $x \in X$, $B_\varepsilon(x)$ will denote the closed ε-ball in X centered at x. For $x \in X$ and $\varepsilon > 0$ define subsets $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$ of $B_\varepsilon(x)$ by $W^s_\varepsilon(x) = \bigcap_{n \geq 0} f^{-n} B_\varepsilon(f^n x)$ and $W^u_\varepsilon(x) = \bigcap_{n \leq 0} f^{-n} B_\varepsilon(f^n x)$. Then we have

(1) $\quad f\, W^s_\varepsilon(x) \subset W^s_\varepsilon(fx)$, $\quad f^{-1} W^u_\varepsilon(x) \subset W^u_\varepsilon(f^{-1}x)$,

(2) $\quad y \in W^\sigma_\varepsilon(x)$ if and only if $x \in W^\sigma_\varepsilon(y)$ $\quad (\sigma = s,u)$,

and

(3) $\quad z \in W^\sigma_{\varepsilon_1+\varepsilon_2}(x)$ when $y \in W^\sigma_{\varepsilon_1}(x)$ and $z \in W^\sigma_{\varepsilon_2}(y)$ $\quad (\sigma = s,u)$.

Definition 1. (X,f) is said to be expansive if there exists a constant $c^* > 0$ such that

$$\{x\} = \bigcap_{n \in \mathbb{Z}} f^{-n}(B_{c^*}(f^n x)) \qquad (= W^s_{c^*}(x) \cap W^u_{c^*}(x))$$

for all $x \in X$, and such a c^* is said to be an expansive constant for f.

For every $\varepsilon > 0$ define subsets Y_ε and Z_ε of $X \times X$ by

$$Y_\varepsilon = \{(x,y) \in X \times X \mid W^s_\varepsilon(x) \cap W^u_\varepsilon(y) \neq \emptyset\}$$

and

$$Z_\varepsilon = \{(x,y) \in X \times X \mid (x,y) \in Y_\varepsilon \text{ and } (y,x) \in Y_\varepsilon\} .$$

For $x \in X$ and $\varepsilon > 0$, subsets $Y_\varepsilon(x)$ and $Z_\varepsilon(x)$ of X are defined by

$$Y_\varepsilon(x) = \{y \in X \mid (x,y) \in Y_\varepsilon\}$$

and

$$Z_\varepsilon(x) = \{y \in X \mid (x,y) \in Z_\varepsilon\} .$$

Then we have

(4) $\quad y \in Z_\varepsilon(x)$ if and only if $x \in Z_\varepsilon(y)$

and

(5) $\quad W^s_\varepsilon(x) \cup W^u_\varepsilon(x) \subset Z_\varepsilon(x) \subset Y_\varepsilon(x)$.

Definition 2. Let $\mathscr{D}$ be a finite partition of X ,i.e., a finite family of subsets in X whose elements are mutually disjoint and $\bigcup_{D \in \mathscr{D}} D = X$. A sequence $\{x_i\}_{i \in \mathbb{Z}}$ of points in X is said to be an α-pseudo orbit with respect to $\mathscr{D}$ if $d(fx_i, x_{i+1}) \leq \alpha$ and $fx_i \underset{\mathscr{D}}{\sim} x_{i+1}$ for all $i \in \mathbb{Z}$ where $x \underset{\mathscr{D}}{\sim} y$ denotes that x and y are in the same element of $\mathscr{D}$. A sequence $\{x_i\}_{i \in \mathbb{Z}}$ of points in X is said to be β-traced if $\bigcap_{i \in \mathbb{Z}} f^{-i}(B_\beta(x_i)) \neq \emptyset$. (X,f) is said to have Takahashi pseudo orbit tracing property (abbrev. TPOTP) if there exists a finite partition $\mathscr{D}$ such that for every $\beta > 0$, there is $\alpha > 0$ such that every α-pseudo orbit with respect to $\mathscr{D}$ is β-traced.

Especially (X,f) is said to have the pseudo orbit tracing property (abbrev. POTP) if $\mathcal{D}$ can be chosen so that $\mathcal{D} = \{X\}$. The notion of TPOTP is firstly used in M. Yuri 39] with a suggestion of Y. Takahashi. It seems likely that for every homeomorphism of a torus, TPOTP implies POTP. However the author does not have the proof.

Let (X,f) be expansive and $c > 0$ be a number such that $2c$ is an expansive constant for f. Then for $(x,y) \in Y_c$, $W_c^s(x) \cap W_c^u(y) \neq \emptyset$ and the set $W_c^s(x) \cap W_c^u(y)$ consists only of one point by expansiveness. Therefore we can define the map $[\ ,\ ] : Y_c \longrightarrow X$ by $(x,y) \longrightarrow [x,y] \in W_c^s(x) \cap W_c^u(y)$ $((x,y) \in Y_c)$, and we have the following;

(6) $\qquad [x,x] = x$,

(7) $\qquad y \in W_c^s(x)$ and $z \in W_c^u(x)$ imply that

$$(y,z) \in Y_c \text{ and } [y,z] = x,$$

(8) $\qquad [x,[y,z]] = [x,z]$ if $(y,z), (x,z) \in Y_c$

$$\text{and } (x,[y,z]) \in Y_c,$$

(9) $\qquad [[x,y],z] = [x,z]$ if $(x,y), (x,z) \in Y_c$

$$\text{and } ([x,y],z) \in Y_c,$$

and

(10) $\qquad W_\varepsilon^s(x) \cap W_\varepsilon^u(y) = \{[x,y]\}$ if $(x,y) \in Y_\varepsilon$ for $\varepsilon \leq c$.

<u>Definition 3.</u> Under the above notations, a subset E of X is said to be a <u>rectangle</u> if $(x,y) \in Y_c$ and $[x,y] \in E$ for all $x,y \in E$.

<u>Definition 4.</u> A family $\mathcal{P}$ of closed rectangles of X is said to be a <u>Markov partition</u> for (X,f) if $\mathcal{P}$ satisfies the following conditions;

1) $P = \overline{\operatorname{int} P}$ for all $P \in \mathcal{P}$,

2) $\bigcup_{P \in \mathcal{P}} P = X$,

3) $\operatorname{int} P \cap \operatorname{int} Q = \emptyset$ for all $P,Q \in \mathcal{P}$ with $P \neq Q$,

4) for every sequence $\{P_n\}_{n \in \mathbb{Z}}$ of elements of $\mathcal{P}$, $\bigcup_{n \in \mathbb{Z}} f^{-n} P_n$ consists at most of one point,

5) $f(W^s_c(x) \cap \operatorname{int} P) \subset W^s_c(fx) \cap \operatorname{int} Q$

and

$$f^{-1}(W^u_c(fx) \cap \operatorname{int} Q) \subset W^u_c(x) \cap \operatorname{int} P$$

whenever $x \in \operatorname{int} P \cap f^{-1}(\operatorname{int} Q)$ $(P,Q \in \mathcal{P})$, and

6) there exist subsets B^s and B^u of X such that $f B^s \subset B^s$, $f^{-1} B^u \subset B^u$, and $B^s \cup B^u = \bigcup_{P \in \mathcal{P}} \partial P$.

Markov partitions are not partitions in strict sence. However we use the word conventionally.

REFERENCES

1] R.L. Adler & B. Weiss. Similarlity of automorphisms of the torus. Mem. Amer. Math. Soc. 98 (1970).

2] N. Aoki. Topological stability of solenoidal automorphisms. Nagoya Math. J. 90 (1983), 119-135 (Correction. ibid. 95 (1984), 103).

3] N. Aoki, M. Dateyama & M. Komuro. Solenoidal automorphisms with specification. Monatsh. Math. 93 (1982), 79-110.

4] K.R. Berg. Convolution and invariant measures, maximal entropy. Math. Syst. Theory 3 (1969), 146-150.

5] R. Bowen. Markov partitions for axiom-A-diffeomorphisms. Amer. J. Math. 92 (1970), 725-747.

6] R. Bowen. Markov partitions and minimal sets for axiom-A-diffeomorphisms. Amer. J. Math. 92 (1970), 907-918.

7] R. Bowen. Periodic points and measures for axiom-A-diffeomorphisms. Trans. Amer. Math. Soc. 154 (1971), 377-397.

8] R. Bowen. Some systems with unique equilibrium states. Math. Syst. Theory. 8 (1974), 193-202.

9] R. Bowen. Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms. Lecture Notes in Math. 470,

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Springer, 1975.

10] R. Bowen Bernoulli equilibrium states for axiom-A-diffeomorphisms. Masth. Syst. Theory 8 (1975), 289-294.

11] R. Bowen. Invariant measures for Markov maps of the interval. Commun. Math. Phys. 69 (1979), 1-17.

12] B.F. Bryant. Expansive self-homeomorphisms of a compact metric space. Amer. Math. Monthly 69 (1962), 386-391.

13] E. Coven & M. Paul. Sofic systems. Israel J. Math. 20, no. 2 (1975), 165-177.

14] M. Dateyama. Invariant measures for homeomorphisms with weak specification. Tokyo J. Math. 4 (1981), 389-397.

15] M. Denker, C. Grillenberger & K. Sigmund. Ergodic Theory on Compact Spaces. Lecture Notes in Math. 527, Springer, 1976.

16] F.T. Farrel & L.E. Jones. Markov cell structures. Bull. Amer. Math. Soc. 83 (1977), 739-740.

17] A. Fathi, F. Laudenbach & V. Poénaru. Travaux de Thurston sur les surfaces (Séminaire Orsey). astérisque 66-67, Soc. Math. France, 1979.

18] M. Gerber & A. Katok. Smooth models of Thurston's pseudo-Anosov maps. Ann. scient. Éc. Norm. Sup., 4^{e} série 15 (1982), 173-204.

19] G.A. Hedlund. Endomorphisms & automorphisms of the dynamical system. Math. Syst. Theory 3 (1969), 320-375.

20] K. Hiraide. On homeomorphisms with Markov partitions. Tokyo J. Math. 8 (1985), 219-229.

21] M.V. Jacobson. On some properties of Markov partitions. Soviet Math. Dokl. 17 (1976), 247-251.

22] M. Komuro. Non-expansive attractors with specification. Tokyo J. Math. 7 (1984), 161-176.

23] M. Komuro. Lorenz attractors donot have the pseudo-orbit tracing property. J. Math. Soc. Japan 37 (1985), 489-514.

24] A. Manning. Axiom A diffeomorphisms have rational zeta functions. Bull. London Math. Soc. 3 (1971), 215-220.

25] W. Parry. Intrinsic Markov chains. Trans. Amer. Math. Soc. 112 (1964), 55-66.

26] W. Parry. An analogue of theprime number theorem for closed orbits of subshifts of finite type and their suspensions. Israel J. Math. 45 (1983), 41-52.

27] W. Parry & M. Pollicott. An analogue of the prime number theorem for closed orbits of Axiom A flows. Ann. of Math. (2), 118 (1983), 573-591.

28] W. Reddy. Expansive canonical coordinates are hyperbolic. Topology Appl. 15, no 2 (1983), 205-210.

29] D. Ruelle. Statistical mechanics on a compact set with $\mathbb{Z}^{\nu}$ action satisfying expansiveness and specification. Trans. Amer. Math. Soc. 185 (1973), 237-253.

30] D. Ruelle. Thermodynamical Formalism - The Mathematical Structure of Classical Equilibrium Statistical Mechanics. Encyclopedia of Math. Appl. 5, Addison-Wesley, 1978.

31] K. Sigmund. Generic properties of invariant measures for Axiom A diffeomorphisms. Invent. Math. 11 (1970), 99-109.

32] K. Sigmund. On mixing measures for axiom A diffeomorphisms. Proc. Amer. Math. Soc. 36 (1972), 497-504.

33] K. Sigmund. On dynamical systems with the specification property. Trans. Amer. Math. Soc. 190 (1974), 285-299.

34] Ja.G. Sinai. Markovian partitions and Y-diffeomorphisms. Funct. Anal. Appl. 2, no 1 (1968), 64-89.

35] Ja.G. Sinai. The construction of Markovian partitions. Funct. Aanl. Appl. 2, no3 (1968), 70-80.

36] Ja.G. Sinai. Gibbs measures in ergodic theory. Russian Math. Surveys 27, no 4 (1972), 21-69.

37] P. Walters. On the pseudo orbit tracing property and its relationship to stability. _In Lecture Notes in Math_. 668, _Springer_, 1978, 231-244.

38] B. Weiss. Subshift of finite type and sofic systems. _Monatsh. Math_. 77 (1973), 462-474.

39] M. Yuri. A construction of an invariant stable foliation by the shadowing lemma. _Tokyo J. Math_. 6 (1983), 291-296.

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